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Fluctuation spectra of few- and large-degree-of-freedom chaotic systems

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Abstract

Following Fujisaka's basic idea, large deviation statistics characterizing temporal fluctuations are calculated based on Mori's projection-operator method. It is shown that this calculation will overcome the finite-sample effect studied by Nakao et al. due to the finiteness of the length of time series, of the time span of the local average, or of the ensemble of numerical data. The proposed method are compared with direct numerical simulations using analytically solvable chaotic maps, and a much wider range of large deviation is observed far beyond the finite-sample direct numerical simulations.

Keywords: large deviation statistics; projection-operator method; fluctuation spectra; rate function; local expansion rates

1. Introduction

It is crucial to extract meaningful information from temporal fluctuations caused by a chaotic dynamical system. Central limit theorem yields a Gaussian probability density function of the finite-time average of a stochastic or a chaotic variable around the long-time average. Characteristic local structures

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of invariant sets of a chaotic dynamical system such as non-hyperbolicity and bifurcations such as tangent bifurcation or crisis may cause large deviations from the long-time average, which cannot be characterized by the central limit theorem.

The large deviation principle [1] is nowadays applied to temporal fluctuations and yields a statistical-thermodynamical formalism originating from a higher-order-moment analysis in the research field of turbulence [2,3]. Especially, large deviations of the local expansion rate or the finite-time Lyapunov exponent have been intensively studied [4,5,6,7].

As a practical problem, a numerical calculation cannot detect rarer events than the numerical rarest event with a probability that a single event in a long but time series. Nakao et al. discussed further this problem called finite-sample effect [8]. In the preceding studies [4,5,6,7], the numerical results are also obtained within this limitation.

Fujisaka proposed a novel calculation method of the two-time correlation function and the large deviation statistics such as rate function based on Mori's projection-operator method, though he did not notice the importance of the finite-sample effect [9, 10]. Mori's method yields an exact closed form of a two-time correlation function of a dynamical variable obeying its equation of motion [11]. The closed form is called a generalized Langevin equation consisting of a systematic term, a memory term and a noise term. Fujisaka's basic idea extends the phase space from the original space, say, (x(t),y(t),z(t)) to the extended one, say, (x(t),x(t+T),x(t+2T),...y(t),y(t+T),y(t+2T),...z(t),z(t+T),z(t+2T),...), or (x(t),x'(t),x''(t),...y(t),y'(t),y''(t),...z(t),z'(t),z''(t),...) and ignores the memory term. The noise term disappears in the closed form of the two-time correlation function due to the orthogonality originating from the projection.

A complicated parameter dependence of diffusion coefficient in a chaotic diffusion is known as fractal diffusion coefficient [12,13]. Its numerical evaluation needs a cumbersome procedure of finding many series of Markov partitions. The above method proposed by Fujisaka is applied to numerical evaluations of fractal diffusion coefficient without finding any Markov partition, which implies that one needs much less numerical effort [14].

In this study, we will show that the above method applied to large deviation calculations reduces numerical efforts and overcomes the finite-sample effect.

2. Large deviation statistics for local expansion rates

Local expansion rates λ_t and the largest Lyapunov exponent Λ_∞ in the case of one-dimensional map $x_{t+1} = f(x_t)$ are respectively given by

$$\lambda_t \equiv \ln \left| \frac{df(x_t)}{dx} \right|, \qquad \Lambda_\infty \equiv \lim_{T\to\infty} \frac{1}{T} \sum_{s=t}^{t+T-1} \lambda_s \,. \tag{1}$$

The positive sign of the largest Lyapunov exponent is often used as a practical existence proof of chaos. The expansion rate, the finite-time average may fluctuate largely in time. Let us assume $\lambda_1, \lambda_2, \cdots$ construct a stationary time series induced by a chaotic map, whose finite-time average $\bar{\lambda}_T$ is given by

$$\bar{\lambda}_T(t) = \frac{1}{T} \sum_{s=t}^{t+T-1} \lambda_s \,. \tag{2}$$

Probability density function $P_T(u)$ of $\bar{\lambda}_T$ is defined as

$$P_T(u) \equiv \lim_{T'\to\infty} \frac{1}{T'} \sum_{t=1}^{T'} \delta\left(u - \bar{\lambda}_T(t)\right) = \langle \delta\left(u - \bar{\lambda}_T(t)\right) \rangle_t \tag{3}$$

Here and hereafter, $\langle \cdots \rangle_t$ denotes the long-time average with respect to time t. Owing to stationarity of λ_t, the law of large number is realized as

$$P_T(u) \to \delta(u - \langle \lambda_t \rangle_t) \quad (T \to \infty) \ . \tag{4}$$

The large deviation principle yields an asymptotic form of $P_T(u)$ for sufficiently-large time interval T as

$$P_T(u) \simeq e^{-S(u)T}, \quad S(u) \geq 0 \ , \tag{5}$$

where $S(u)$ is called fluctuation spectra or rate function and depends only on u, which can be regarded as a convergence rate of $P_T(u)$ to the above mentioned delta function implying the law of large number. Then, generating function is defined as

$$Z_q(T) \equiv \langle e^{qT\bar{x}_T(t)} \rangle_t = \int_{-\infty}^{\infty} e^{qTu} P_T(u) \, du \ . \tag{6}$$

Substituting Eq. (5) into Eq. (6), we have

$$Z_q(T) = \int_{-\infty}^{\infty} e^{[qu - S(u)]T} \, du \ , \tag{7}$$

which leads to the following expression by use of a saddle-point calculation:

$$Z_q(T) \simeq e^{[qu^* - S(u^*)]T} \ . \tag{8}$$

Here, u^* denotes u maximizing $qu - S(u)$. Since $T\bar{u}_T$ is proportionate to T on average for sufficiently-large T, there exist $\phi(q)$ to satisfy

$$Z_q(T) \simeq e^{\phi(q)T} \ . \tag{9}$$

Therefore, the relation

$$\phi(q) = qu^* - S(u^*) = \max_u [qu - S(u)] \tag{10}$$

is obtained and considered as a Legendre transform between $S(u)$ and $\phi(q)$. Thus, fluctuation spectra $S(u)$ is obtain as

$$S(u) = \max_q [qu - \phi(q)] = q\frac{d\phi(q)}{dq} - \phi(q) \ . \tag{11}$$

3. Calculation method of fluctuation spectra based on Mori's projection-operator formalism

Except for a few exactly solvable models, the averaging procedure described in the preceding section causes the finite-sample effect in numerical calculations. In this section, we explain a novel calculation method based on Mori's projection-operator formalism following Fujisaka's basic idea.

Introducing a new variable $s_t(\tau)$ satisfying

$$s_t(\tau) \equiv L_q^t 1 = \exp\left(q \sum_{s=\tau}^{\tau+t-1} \lambda_s \right) \tag{12}$$

we can rewrite the generating function in the form of a two-time correlation function as

$$Z_q(t) \equiv \langle e^{qt\bar{\lambda}_t(\tau)} \rangle_\tau = \langle s_t(\tau) \rangle_\tau = \langle s_t(\tau) s_0(\tau) \rangle_\tau \,. \tag{13}$$

Starting form the time-evolution operator of $s_t(\tau)$, we obtain its two-time correlation function in the following closed form based on Mori's projection-operator method:

$$\langle s_{t+1}(\tau) s_0(\tau) \rangle_\tau = \zeta_q \langle s_t(\tau) s_0(\tau) \rangle_\tau + \sum_{k=0}^{t-1} \Psi_q(t-1-k) \langle s_k(\tau) s_0(\tau) \rangle_\tau \,. \tag{14}$$

Then, extending from the original one-dimensional trajectory $s_t(\tau)$ to the delayed coordinate

$$\boldsymbol{s}_t(\tau) \equiv (s_t(\tau), s_{t+1}(\tau), \ldots, s_{t+m}(\tau))^{\mathrm{T}} \,, \tag{15}$$

and ignoring the memory term, we have the following relation for the extended variable $\boldsymbol{s}_t(\tau)$:

$$\langle \boldsymbol{s}_{t+1}(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau \simeq \hat{\zeta}_q \langle \boldsymbol{s}_t(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau, \qquad \hat{\zeta}_q \equiv \langle \boldsymbol{s}_1(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau \langle \boldsymbol{s}_0(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau^{-1}, \tag{16}$$

which can be straightforwardly solved as

$$\langle \boldsymbol{s}_t(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau \simeq \left(\hat{\zeta}_q\right)^t \langle \boldsymbol{s}_0(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau = \left\{ \langle \boldsymbol{s}_1(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau \langle \boldsymbol{s}_0(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau^{-1} \right\}^t \langle \boldsymbol{s}_0(\tau) \boldsymbol{s}_0(\tau)^{\mathrm{T}} \rangle_\tau \,. \tag{17}$$

Note that (1, 1) element of the matrix appearing in the left-hand side is equal to the original generating function of concern.

4. Comparison between the proposed method and direct numerical simulation

There exist some solvable models yielding an analytical expression of the rate function of the local expansion rate, or the fluctuation spectrum, for short. In this section, we compare between the results obtained from the analytical treatment, from the proposed method and from direct numerical simulations or two solvable chaotic maps.

4.1. Fully-developed asymmetric tent map

A fully-developed asymmetric tent map is given by

$$x_{t+1} = \begin{cases} 4x_t & 0 \le x_t < \frac{1}{4} \;, \\ \frac{4}{3}(1-x_t) & \frac{1}{4} \le x_t \le 1 \;, \end{cases} \tag{18}$$

whose fluctuation spectrum is given by

$$S(u) = u - \frac{u - u_{min}}{u_{max} - u_{min}} \ln\left(\frac{u_{max} - u_{min}}{u - u_{min}}\right) - \frac{u_{max} - u}{u_{max} - u_{min}} \ln\left(\frac{u_{max} - u_{min}}{u_{max} - u}\right) \,,$$
$$u_{max} = \ln 4, \quad u_{min} = \ln\frac{4}{3} \;. \tag{19}$$

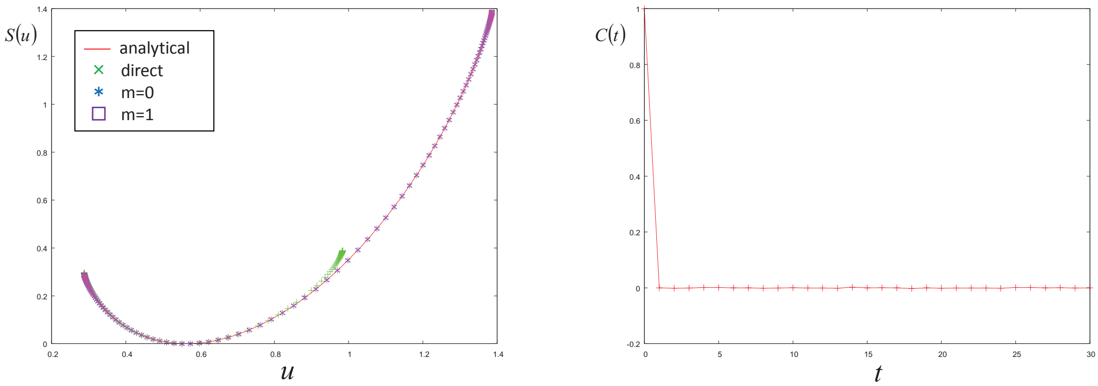

Fig. 1. Fully-developed asymmetric tent map. (a) Fluctuation spectra. (b) Two-time correlation function of local expansion rate.

Figure 1(a) depicts fluctuation spectra, in which magenta, red, green, blue symbols correspond respectively to the analytical expression, our method only ignoring the memory term, our method with two-dimensional extended phase space, and numerical evaluations according the definition of the fluctuation spectrum. The finite-sample effect appears so clearly that blue symbols trace only the part of the exact fluctuation spectrum. As shown in Fig.1(b), the two-time correlation function is nothing but the Kronecker delta, so that only the first or the second step of our approximation is enough to approach the exact result, which is a peculiarity of this model.

4.2. Fully-developed logistic map

For the fully-developed logistic map $x_{t+1} = 4x_t(1-x_t)$, the fluctuation spectrum is obtained as the following piecewise linear form:

$$S(u) = \begin{cases} \Lambda_\infty - u & (u \le \Lambda_\infty) \quad , \\ u - \Lambda_\infty & (\Lambda_\infty \le u \le 2\Lambda_\infty) \quad , \\ \infty & (u > 2\Lambda_\infty) \quad , \end{cases} \tag{20}$$

where the largest Lyapunov exponent is given by $\Lambda_\infty = \ln 2$.

Figure 2(a) depicts fluctuation spectra, in which the analytical expression and numerical evaluations according the definition of the fluctuation spectrum are drawn respectively with a red line and green symbols. It is easy to observe that the latter traces only a very narrow range around the largest Lyapunov exponent. Our method only ignoring the memory term is shown as blue symbols. Other symbols correspond to our method with two-dimensional to six-dimensional extended phase space. An enlarged figure around the largest Lyapunov exponent is shown in Fig.2(b). Unlike the preceding subsection, the way of convergence to the analytical result is not simple, since the two-time correlation function has a characteristic structure and yields a finite correlation time, as shown Fig.3. Our method also needs an ensemble of numerical data, which causes another kind of finite-sample effect. The further the approximation step proceeds, the narrower the large deviation is observed. However, the curvature approaches to a straight line, as the step proceeds further. Comparison of green symbols with other symbols implies that our method is a very effective calculation method of large deviations.

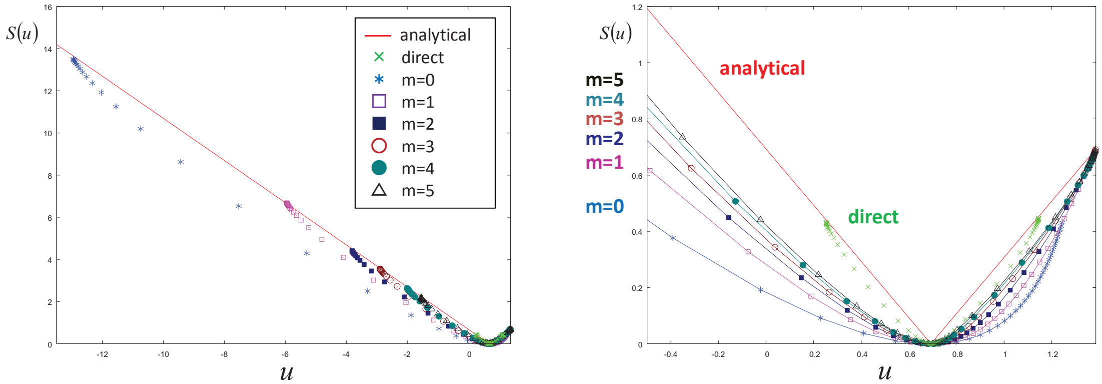

Fig. 2. Fully-developed logistic map. (a) Fluctuation spectra. (b) Enlarged figure around the largest Lyapunov exponent.

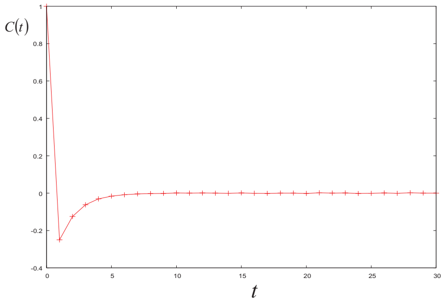

Fig. 3. Two-time correlation function of the local expansion rate of the fully-developed logistic map.

5. Concluding remarks

Using two solvable chaotic maps, we show that direct numerical simulations according to the definition of the fluctuation spectrum trace only narrow part of the exact spectrum due to the finite-sample effect and that our method yields a much wider range of the spectrum. It is a future problem that our method is verified for general chaotic dynamical systems.

Though fluctuation spectra for large degree-of-freedom chaotic systems have many interesting aspects, we omit this topic in this manuscript due to the limited space.

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